Eigenmodes and Basis Functions for Pitch-Angle Diffusion with a Partially Filled Loss Cone

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Pitch-Angle Diffusion Equation

$$\frac{d}{dz}\left[D_{zz}\frac{dg_n}{dz}\right] + \lambda_n g_n$$

$$= \begin{cases} 0, & z < z_c \\ (\Omega_2/\pi)(1-\eta)g_n, & z > z_c \end{cases}$$

Definitions

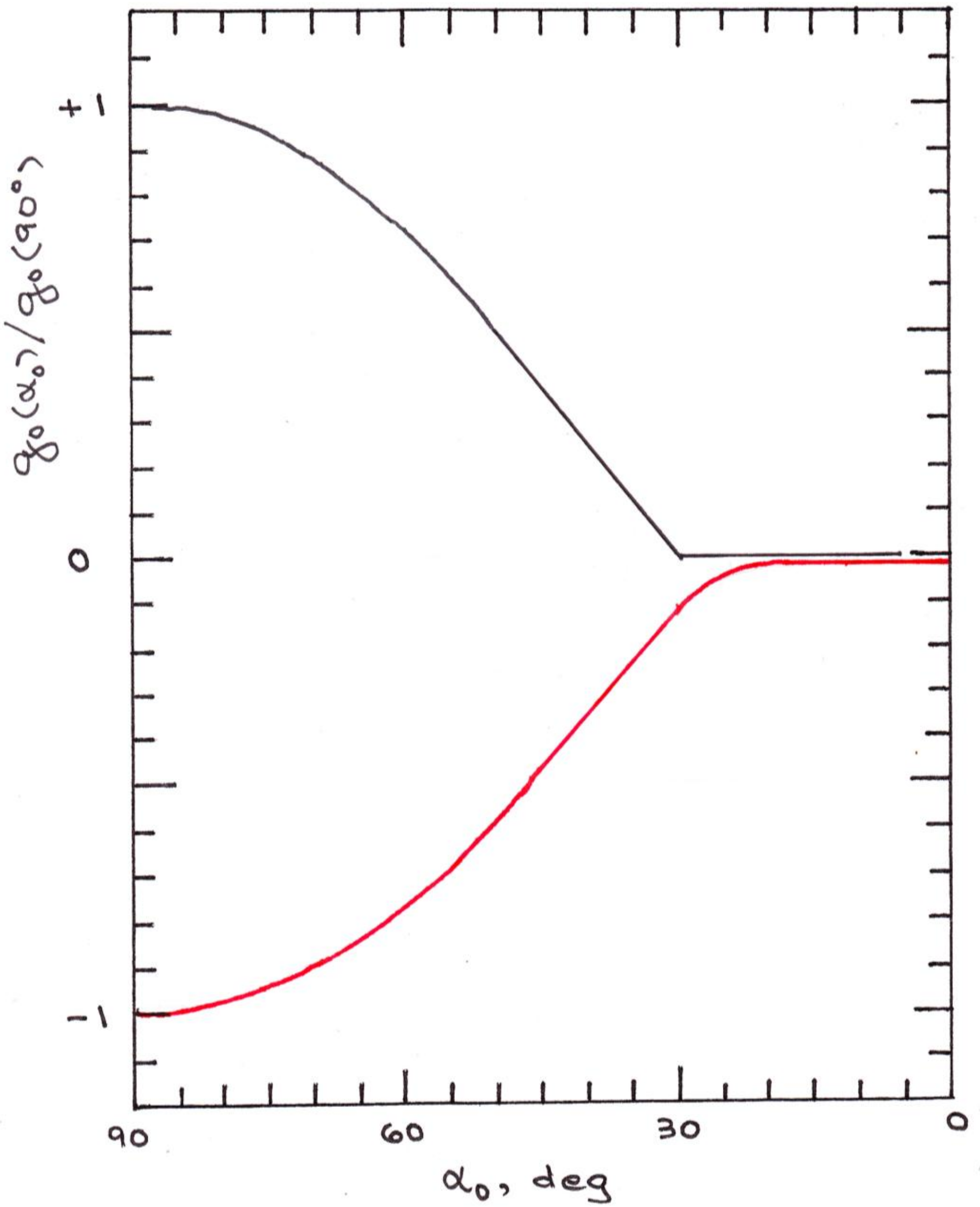
$$y \equiv \sin \alpha_0$$

$$z = Z(y) = \int_y^1 y' T(y') dy'$$

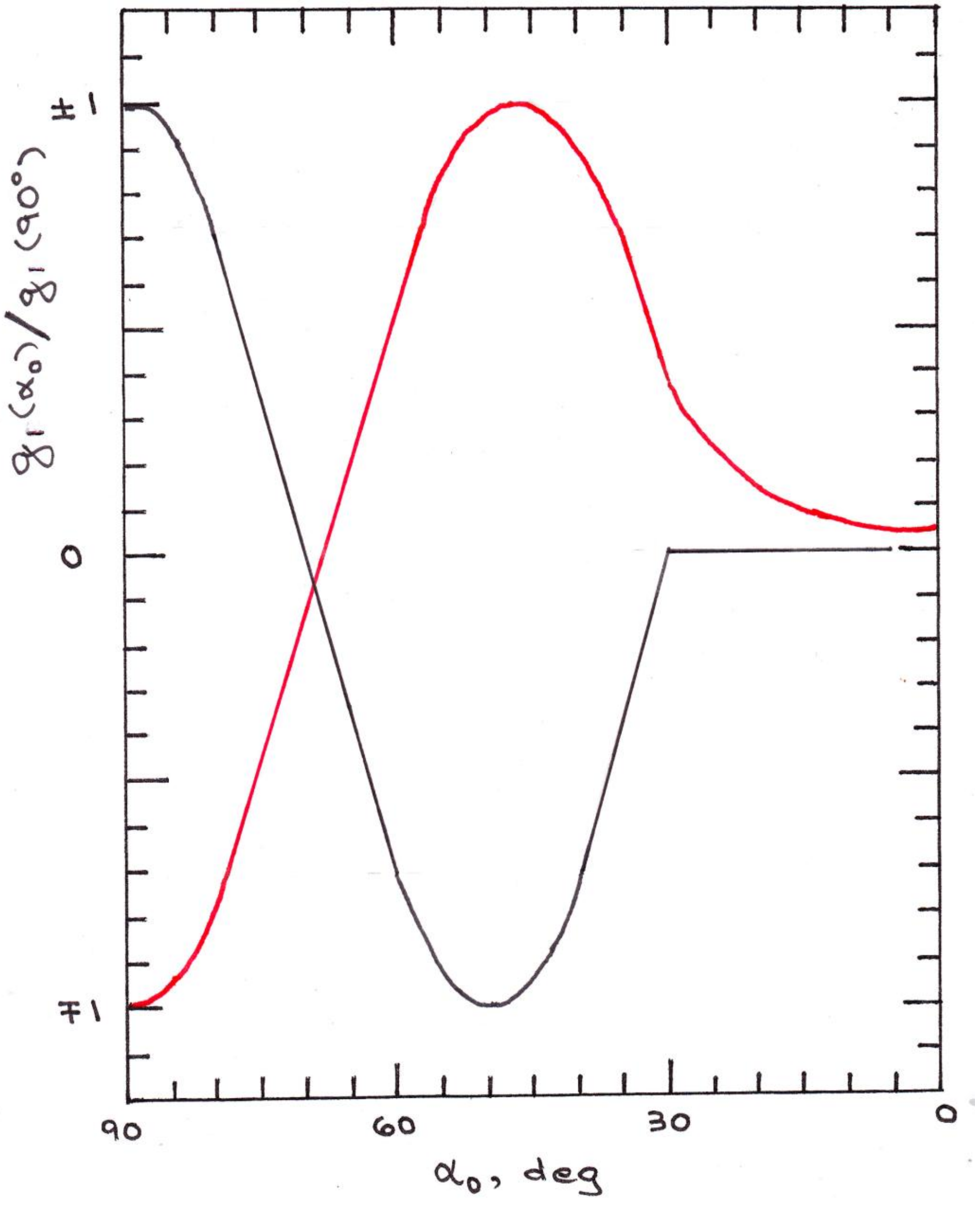
$$\approx 0.6901(1 - y^2) - 0.2326(1 - y^{11/4})$$

$$T(y) = (v/4La)(2\pi/\Omega_2)$$

$$\approx 1.3802 - 0.6397 y^{3/4}$$



+1
0
-1
$g_0(\alpha_0)/g_0(90°)$
90
60
30
0
α_0, deg



$g_1(\alpha_0)/g_1(90°)$
±1
0
∓1
90
60
30
0
α_0, deg

Solutions for Constant D_{zz}

$$g_n(z) \propto \cos[(\lambda_n/D_{zz})^{1/2} z]\,, \quad z < z_c$$

$$g_n(z) \propto \cosh[k_n(z - z_0)]\,, \quad z > z_c$$

$$z_0 \equiv Z(0) = 16/35$$

$$k_n^2 = \frac{(1-\eta)\langle\Omega_2\rangle}{\pi D_{zz}} - \frac{\lambda_n}{D_{zz}}$$

$$k_n \tanh[k_n(z_0 - z_c)]$$

$$= (\lambda_n/D_{zz})^{1/2} \tan[(\lambda_n/D_{zz})^{1/2} z_c]$$

$$(\lambda_n/D_{zz})^{1/2} =$$

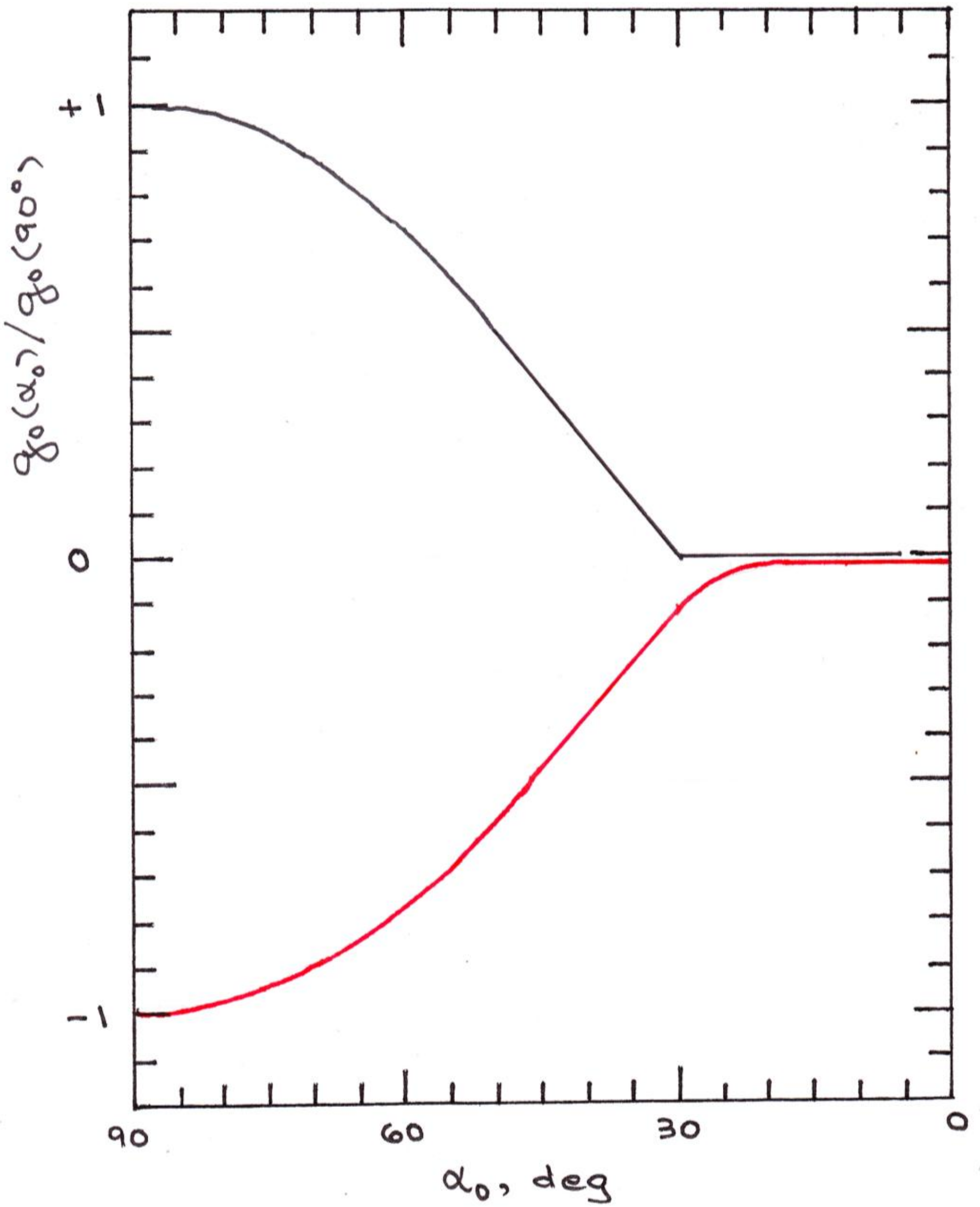
$$(1/z_c) \tan^{-1}\{(\lambda_n/D_{zz})^{-1/2} k_n \tanh[k_n(z_0 - z_c)]\}$$

Table 4. Directional flux ratios for lowest eigenmode ($\alpha_c = 5°$).

$\frac{\pi D_{zz}}{(1-\eta)\langle\Omega_2\rangle}$	flux ratio : $\alpha_0 = 0$ vs $\alpha_0 = 90°$	flux ratio : $\alpha_0 = 0$ vs $\alpha_0 = \alpha_c$
10^{-6}	0.000048655	0.014045975
10^{-5}	0.004748619	0.399582996
10^{-4}	0.064037320	0.888624184
10^{-3}	0.442028147	0.987903180
10^{-2}	0.897446030	0.998784268
10^{-1}	0.988884150	0.999878399
10^{0}	0.998879136	0.999987839

Table 1. Solutions of (7b) for various normalized diffusion coefficients ($\alpha_c = 5°$).

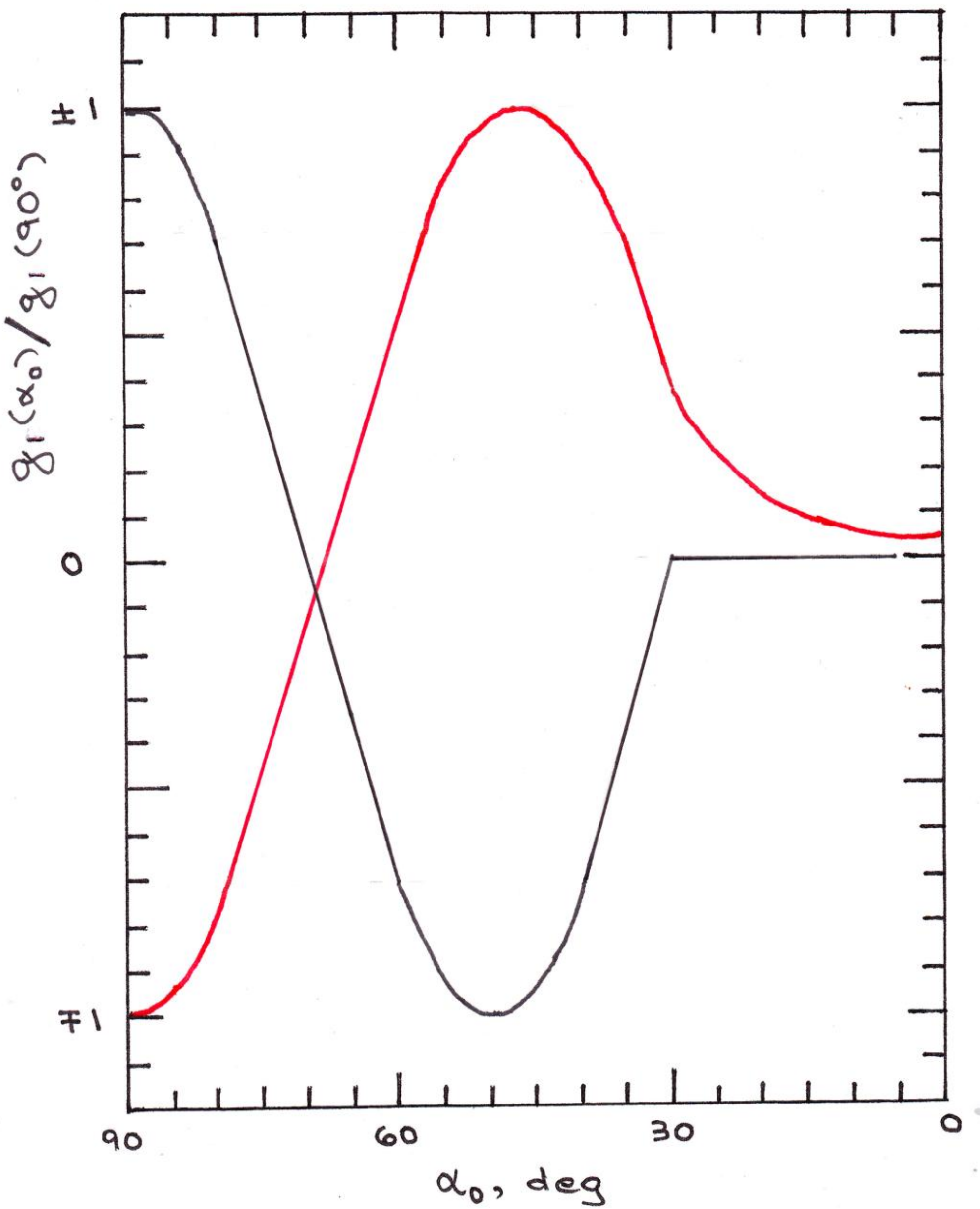
$\frac{\pi D_{zz}}{(1-\eta)\langle\Omega_2\rangle}$	$\frac{\text{numerical result}}{\text{for } k_0}$	$\frac{\lambda_0}{D_{zz}} = \frac{\text{electron loss rate}}{\text{diffusion coeff}}$
0	∞	$12.04976577 = (\pi/2z_c)^2$
10^{-6}	999.9940016	11.99667930
10^{-5}	316.2090003	11.86812502
10^{-4}	99.94514197	10.96859721
10^{-3}	31.52802804	5.983447552
10^{-2}	9.949461317	1.008219505
10^{-1}	3.145219678	0.107593176
10^{0}	0.994569743	0.010831026



$g_0(\alpha_0)/g_0(90°)$
+1
0
-1
90
60
30
0
α_0, deg

Table 2. Solutions of (7b) for various normalized diffusion coefficients ($\alpha_c = 5°$).

$\dfrac{\pi D_{zz}}{(1-\eta)\langle\Omega_2\rangle}$	$\dfrac{\text{numerical result}}{\text{for } k_1}$	$\dfrac{\lambda_1}{D_{zz}} = \dfrac{\text{electron loss rate}}{\text{diffusion coeff}}$
0	∞	$108.4478919 = (3\pi/2z_c)^2$
10^{-6}	999.9460135	107.9701060
10^{-5}	316.0588351	106.8127568
10^{-4}	99.50493049	98.76880927
10^{-3}	30.57164174	65.37472138
0.002	21.04224439	57.22395114
0.005	12.19058992	51.38951740
0.010	7.120187025	49.30293673
0.020	1.327525081	48.23767716
$(z_c/\pi)^2$	0.000000000	48.19906305



±1
0
∓1
90
60
30
0
g₁(α₀)/g₁(90°)
α₀, deg

Table 3. Solutions of (7b) for various normalized diffusion coefficients ($\alpha_c = 5°$).

$\frac{\pi D_{zz}}{(1-\eta)\langle\Omega_2\rangle}$	numerical result for k_2	$\frac{\lambda_2}{D_{zz}} = \frac{\text{electron loss rate}}{\text{diffusion coeff}}$
0	∞	$301.2441440 = (5\pi/2z_c)^2$
10^{-6}	999.8500303	299.9169187
10^{-5}	315.7582936	296.7000531
10^{-4}	98.61727534	274.6330043
10^{-3}	28.11979514	209.2771213
0.002	17.34092632	199.2922744
0.005	2.655051431	192.9507019
$(z_c/2\pi)^2$	0.000000000	192.7962522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Empty Loss Cone (Weak Pitch-Angle Diffusion)

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[D_{zz}\frac{\mathrm{d}g_n}{\mathrm{d}z}\right] + \lambda_n g_n = 0$$

$$y \equiv \sin\alpha_0$$

$$z = Z(y) = \int_y^1 y'\mathrm{T}(y')\mathrm{d}y'$$

$$\approx 0.6901(1-y^2) - 0.2326(1-y^{11/4})$$

Expansion in Basis Functions

$$g_n(z) \approx \sum_m \hat{g}_m(z) U_{mn}$$

$$\hat{g}_n(z) = (2\theta_n/\zeta_{\nu n})^{1/2}\,[J_\nu'(\zeta_{\nu n})]^{-1}\,J_\nu(\theta_n)$$

$$\times\, D_{zz}^{-1/4}\Big[\int_0^{z_c} D_{zz}^{-1/2}\mathrm{d}z\Big]^{-1/2}$$

$$\theta_n = \zeta_{\nu n}\Big[\int_0^{z} D_{z'z'}^{-1/2}\mathrm{d}z'\Big]^{-1/2}$$

$$\div\,\Big[\int_0^{z_c} D_{z'z'}^{-1/2}\mathrm{d}z'\Big]^{-1/2}$$

$$J_\nu(\zeta_{\nu n}) = 0 \quad (n = 0, 1, 2, \ldots)$$

$$\nu = (\sigma - 1)/(2 - \sigma)$$

$$\sigma = \lim_{z \to 0} (\mathrm{d} \ln D_{zz} / \mathrm{d} \ln z)$$

$$\zeta_{\nu n} \approx n\pi + (2\nu + 3)(\pi/4)$$

$$g_n(z) \approx \sum_m \hat{g}_m(z) U_{mn}$$

$$\Lambda_{nm} \equiv \int_0^{z_C} (\mathrm{d}\hat{g}_n / \mathrm{dz}) D_{zz} (\mathrm{d}\hat{g}_m / \mathrm{dz}) \mathrm{dz}$$

$$\lambda_n \approx \Lambda_{nn} - \sum_{k \neq n} \frac{\Lambda_{nk}\Lambda_{kn}}{\Lambda_{kk} - \Lambda_{nn}}$$

$$\frac{U_{kn}}{U_{nn}} \approx \frac{1}{\Lambda_{nn} - \Lambda_{kk}} \left[\Lambda_{kn} + \sum_{j \neq k,n} \frac{\Lambda_{kj} \Lambda_{jn}}{\Lambda_{nn} - \Lambda_{jj}} \right]$$

$$U_{nn} \approx [1 + \sum_{k \neq n} (U_{kn} / U_{nn})^2]^{-1/2}$$


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